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ON NONLINEAR BEAM MODELS FROM THE POINT OF VIEW OF COMPUTATIONAL POST-BUCKLING ANALYSIS

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Abstract—The buckling and post-buckling analysis of elastic planar frames is considered and the use of geometrically exact beam models is thereby advocated. It is shown that usual technical beam models fail to predict correctly the curvature of the post-buckling curve at bifurcation even for standard problems of elastic stability theory. It is also argued that versatile and efficient computational procedures for bifurcation analysis of general planar frames are to be based on unconstrained beam models. Some remarks on finite element representation of nonlinear beam models are passed in conclusion.

1. INTRODUCTION

The present paper—devoted to geometrically nonlinear planar beam models—is oriented toward the post-buckling analysis of beams and frames, within the scope of Koiter's theory of elastic stability[1]. This theory—while widely successful as a rational explanation of the observable nonlinear behaviour of elastic structures—has not produced so far a huge number of specific, quantitative results. In particular, the perturbation procedures deriving from Koiter's approach have not yet gained an established position within the field of computational mechanics (as properly defined by Oden and Bathe[2]). From this point of view, the overall situation has not changed very much, since the appearance of the tepid review article by Gallagher in 1975[3].

Nevertheless, the writers maintain that efficient and versatile numerical procedures for post-buckling analysis could actually be implemented—provided the Koiter's theory be given a suitable computational setting, and the relevant information on nonlinear behaviour be properly embodied within the computer program. In a preceding paper[4], this opinion has been discussed at large and some general conclusions about the feasibility of a computer-oriented perturbation method have been drawn. One of them—most pertinent to the present subject—is that *a tractable compatible model can not encompass nonlinear kinematical constraints* (the basic motivation for this is being given in Section 2). We are hence confronted with the task of considering beam models not incorporating axial and/or shear undeformability constraints, which would unavoidably be nonlinear—if enforced with the accuracy required. Two alternatives are described in Section 3.

Geometrically exact nonlinear beam models—while obviously far from new (see, e.g. the treatise by Antman[5]) have not been widely employed in structural mechanics. On the contrary, "a great number of nonlinear technical theories (have been developed) that are based upon *ad hoc* assumptions leading to the neglect of certain kinematical terms regarded as small. These theories enjoy neither the accuracy and the generality of the complete nonlinear theory nor the analytic simplicity of the linear theory" (Antman[5] Section 1). On our part, we may add that technical theories prove to be unreliable in predicting the post-buckling solution of even the most standard beam problems (evidence is to be found in Section 6). Hence, the warning set forth by Koiter[6] against the omission of "small" terms within the equations of elastic stability for thin shells, has to be repeated also in this comparatively simpler context (at least as far as *post*-buckling behaviour is concerned).

The bulk of theoretical, as well as experimental, research on post-buckling of beams and frames has been done in the United Kindgom; a comprehensive account of this work (up to 1973) is given in the book by Britvec[7]. Throughout this volume the inextensible, shear undeformable beam model is nearly invariably considered. The same model is used in the book by Thompson and Hunt[8] which undoubtedly is possessed of a more pronounced computational flavour. The finite element solutions presented by these authors for two sample problems (the Euler strut and the Roorda frame), notwithstanding their good convergence properties, clearly demonstrate that the finite element fabrication of *finite* displacement fields for internally constrained beams can hardly be contrived into a general automatic procedure (see Section 7).

2. A SUMMARY OF BUCKLING AND POST-BUCKLING ANALYSIS

In this section we briefly sketch the fundamentals of bifurcation analysis developed in Ref. [4], broadly following the treatment by Budiansky[9] and his compendious notation. Before entering the matter, two remarks are due: (i) here—as well as in Ref. [4]—only simple bifurcation is considered, and (ii) no mention is made here of the treatment of so-called "imperfect" systems—in contrast to Ref. [4], which hinges upon this topic; in fact, the very idea of a computationally sound perturbation procedure depends on the *automatic* fabrication of a suitable "perfect" system, and the relevant "imperfection term", starting from a given imperfect system. However, there is nothing peculiar to beam models to be discussed in this connection; the interested reader is therefore referred to our paper[4].

We shall consider perfect (i.e. bifurcating) systems characterized by a total potential energy functional

$$\Pi(u, \lambda) \overset{\text{def}}{=} \Phi(u) - f(\lambda)u \tag{2.1}$$

u being the displacement field of the hyper-elastic body under consideration and λ a parameter governing the external force field acting on it (for simplicity, f is supposed to be independent of u). Functional $\Phi(\cdot)$ associates to each displacement field, belonging to a suitable function space $\mathscr{V}$ the corresponding value of the elastic strain energy.

The equilibrium condition is obtained by requiring the functional $\Pi(\cdot, \lambda)$ to be stationary *within the set $\mathscr{U} \subset \mathscr{V}$ of kinematically admissible displacement fields*, i.e.

$$\Pi'(u, \lambda)\delta u \equiv \Phi'(u)\delta u - f(\lambda)\delta u = 0, \; \forall \delta u \in \mathscr{T}(u) \tag{2.2}$$

where $\mathscr{T}(u) \subset \mathscr{V}$ is the tangent space to $\mathscr{U}$ in u, and a prime denotes (Fréchet) differentiation with respect to u in $\mathscr{V}$. Equation (2.2) implicitly establishes a relationship between pairs (u, λ) which make it satisfied. We shall call ***equilibrium path*** a smooth branch of such a relationship, and presume that *two* distinct equilibrium paths cross each other at a bifurcation—or "critical"—point (u_c, λ_c). One of them (named *fundamental* path) is supposed to be known in advance, in the explicit form

$$u = u^f(\lambda) \tag{2.3}$$

and we intend to determine the second one (*bifurcated* path) in a neighbourhood of the bifurcation point. The bifurcated path will be described in the parametric form

$$\lambda = \lambda(t) \tag{2.4^1}$$

$$u = u^b(t) \tag{2.4^2}$$

t representing a suitably defined abscissa along the path. Obviously, the fundamental path (2.3) is also susceptible of the parametric representation, companion to eqns (2.4)

$$\lambda = \lambda(t) \tag{2.5^1}$$

$$u = u^f(\lambda(t)) \tag{2.5^2}$$

(notice that eqns (2.4[1]) and (2.5[1]) coincide). It proves useful to introduce next the *differential displacement* from the fundamental to the bifurcated path for a given value of t:

$$v(t) \overset{\text{def}}{=} u^b(t) - u^f(\lambda(t)). \tag{2.6}$$

At bifurcation, the differential displacement vanishes by definition. In the following, we shall make the null value of t to correspond to bifurcation; hence

$$v(0) = 0. \tag{2.7}$$

By definition, the equilibrium equation (2.2) is *identically* satisfied along both paths (2.4), (2.5), i.e.

$$\Pi'(u^f(\lambda(t)), \lambda(t))\delta u = 0, \ \forall \delta u \in \mathcal{T}(u^f(\lambda(t))) \qquad (2.8^1)$$
$$\Pi'(u^b(t), \lambda(t))\delta u = 0, \ \forall \delta u \in \mathcal{T}(u^b(t)) \qquad (2.8^2)$$

for *any* value of t (not too far from $t=0$). Note that in general, the spaces of virtual displacements $\mathcal{T}$ appearing within eqns (2.8) do depend on t. Obviously, this circumstance creates extreme difficulties when one wants to differentiate eqns (2.8) with respect to t. If, and only if, the set $\mathcal{U}$ of kinematically admissible displacement fields is a *linear manifold* in $\mathcal{V}$, i.e.

$$\mathcal{U} = u_0 + \mathcal{W} \tag{2.9}$$

$\mathcal{W}$ being a subspace of $\mathcal{V}$, $\mathcal{T}(u)$ turns out to be independent of u:

$$\mathcal{T}(u) = \mathcal{W}, \quad \forall u \in \mathcal{U}. \tag{2.10}$$

In this case, the kth t-derivatives of eqns (2.8) are straightforwardly evaluated at $t=0$:

$$\left.\left.\begin{aligned} &\overset{(k)}{\Pi'}(u^f(\lambda(t)), \lambda(t))\Big|_{t=0} \delta u = 0 \qquad (2.11^1)\\ &\overset{(k)}{\Pi'}(u^b(t), \lambda(t)\Big|_{t=0} \delta u = 0 \qquad (2.11^2)\end{aligned}\right\} \forall \delta u \in \mathcal{W}.\right.$$

As mentioned in Section 1, this is the basic reason why nonlinear kinematic constraints—which would invalidate eqn (2.9)—should be avoided (if a compatible model is sought). Obvious as it is, it is worthy of mention that difficulties of this kind simply disappear when dealing with linear problems.

Equations (2.11) constitute the starting-point of our perturbation analysis. By subtracting the first t-derivatives along the two paths (by correspondence with $t=0$), we get

$$\Phi''(u^f(\lambda_c))\dot{v}_c\delta u = 0, \ \forall \delta u \in \mathcal{W} \tag{2.12}$$

where a superposed dot denotes differentiation with respect to t (i.e. $\dot{v} \equiv \overset{(1)}{v}$), and suffix c labels quantities evaluated at bifurcation ($t=0$). The solution of the eigenvalue problem (2.12) yields both the *critical load parameter* λ_c and the *buckling mode* $\dot{v}_c$ (which is supposed to be simple). This concludes the linearized buckling analysis; the post-buckling analysis is performed by pair-wise subtracting higher-order t-derivatives (2.11). From the second derivatives ($k=2$) we get

$$\{\Phi''_c\ddot{v}_c + \Phi'''_c\dot{v}_c^2 + 2\dot{\lambda}_c\Phi'''_c\hat{u}_c^f\dot{v}_c\}\delta u = 0, \ \forall \delta u \in \mathcal{W} \tag{2.13}$$

where a hat denotes differentiation with respect to λ. From the third derivatives ($k=3$) we get

$$\begin{aligned}\{&\Phi''_c\dddot{v}_c + \Phi_c^{IV}\dot{v}_c^3 + 3\dot{\lambda}_c\Phi_c^{IV}\hat{u}_c^f\dot{v}_c^2 + 3\dot{\lambda}_c^2\Phi_c^{IV}[\hat{u}_c^f]^2\dot{v}_c + 3\Phi'''_c\dot{v}_c\ddot{v}_c\\ &+ 3\dot{\lambda}_c\Phi'''_c\hat{u}_c^f\ddot{v}_c + 3\dot{\lambda}_c^2\Phi'''_c\hat{\hat{u}}_c^f\dot{v}_c + 3\ddot{\lambda}_c\Phi'''_c\hat{u}_c^f\dot{v}_c\}\delta u = 0, \ \forall \delta u \in \mathcal{W}\end{aligned} \tag{2.14}$$

and so on, for $k > 3$. While more and more exacting, the procedure is evidently modular. Notice in particular that the same *self-adjoint singular* operator Φ_c'' appears within eqns (2.13) and (2.14), which may be regarded as (linear) equations in $\ddot{v}_c$ and $\dddot{v}_c$, respectively, to be solved in succession. The enforcement of *Fredholm orthogonality condition* on eqns (2.13), (2.14) delivers the value of $\dot{\lambda}_c$ and $\ddot{\lambda}_c$, respectively:

$$\dot{\lambda}_c = -\frac{\Phi_c''' \dot{v}_c^3}{2\Phi_c''' \hat{u}_c^f \dot{v}_c^2} \tag{2.15^1}$$

$$\ddot{\lambda}_c = -\frac{\Phi_c^{IV}\{\dot{v}_c^4 + 3\dot{\lambda}_c \hat{u}_c^f \dot{v}_c^3 + 3\dot{\lambda}_c^2 [\hat{u}_c^f]^2 \dot{v}_c^2\}}{3\Phi_c''' \hat{u}_c^f \dot{v}_c^2} - \frac{3\Phi_c'''\{\dot{v}_c^2 \ddot{v}_c + \dot{\lambda}_c \hat{u}_c^f \dot{v}_c \ddot{v}_c + \dot{\lambda}_c^2 \hat{\dot{u}}_c^f \dot{v}_c^2\}}{3\Phi_c''' \hat{u}_c^f \dot{v}_c^2}. \tag{2.15^2}$$

In conclusion, the results of the above described perturbation analysis furnish the first n terms of the series expansions

$$\lambda = \lambda_c + \sum_{k=1}^{n} \frac{1}{k!} \overset{(k)}{\lambda}_c t^k + o(t^n) \tag{2.16^1}$$

$$v = \sum_{k=1}^{n} \frac{1}{k!} \overset{(k)}{v}_c t^k + o(t^n) \tag{2.16^2}$$

which—under suitable smoothness conditions—may be used for representing the bifurcated path near (u_c, λ_c). We shall truncate series (2.16) at $n = 2$, without pursuing the evaluation of higher order terms.

3. KINEMATICS OF PLANAR BEAMS

Beam theory may be developed either from the three-dimensional theory, or directly as a one-dimensional model. Subsections 3.1 and 3.2 are devoted to these two distinct approaches, respectively. The simplest kinematical setting will be adopted in both.

The line of centroids of the beam will be assumed to be *straight* in its *reference configuration*, and the cross-sections to be *normal* to this straight line. An orthonormal basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for the ambient space $\mathfrak{E}^3$ will be selected, in such a way that the line of centroids—in its reference configuration—lies onto the interval $[0, \ell]$ of the x-axis (with $\ell > 0$). The abscissa $s \in [0, \ell]$ will be used as a material coordinate, labelling the cross-section whose centroid has reference position $(s, 0, 0)$.

The generic configuration of the beam, undergoing a deformation parallel to the plane spanned by $\mathbf{i}$, $\mathbf{j}$, will be identified through the vector function

$$\mathbf{u}(\cdot) = u(\cdot)\mathbf{i} + v(\cdot)\mathbf{j} \tag{3.1}$$

and the scalar function $\phi(\cdot)$, $\mathbf{u}(s)$ as representing the *displacement* of the centroid s and $\phi(s)$ as measuring the *rotation* of the cross-section s (about the z-axis). The *position vector* of a centroid $\mathbf{r}$ is hence connected to $\mathbf{u}$ through

$$\mathbf{r}(s) = s\mathbf{i} + \mathbf{u}(s) \tag{3.2}$$

while the *unit normal vector* $\mathbf{a}$ to a cross-section depends on ϕ as follows

$$\mathbf{a}(s) = \cos\phi(s)\mathbf{i} + \sin\phi(s)\mathbf{j} \tag{3.3}$$

(in the reference configuration $\phi(s) \equiv 0 \Rightarrow \mathbf{a}(s) \equiv \mathbf{i}$).

3.1 *Development from the three-dimensional theory*

We have first to obtain a three-dimensional displacement field

$$\mathbf{u}^P(x, y, z) = u^P(x, y, z)\mathbf{i} + v^P(x, y, z)\mathbf{j} + w^P(x, y, z)\mathbf{k} \tag{3.4}$$

depending on the previously introduced one-dimensional fields $\mathbf{u}(s)$, $\phi(s)$. It is intended that (x, y, z) are the coordinates of the *reference* position of a body-point.

A *planar* displacement field $\mathbf{u}^P(x, y, z)$ will be constructed as follows (see Fig. 1)

$$\begin{aligned} u^P(x, y, z) &= u(x) - y \sin \phi(x) \\ v^P(x, y, z) &= v(x) + y[\cos \phi(x) - 1] \\ w^P(x, y, z) &= 0 \end{aligned} \tag{3.5}$$

and the Green tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{h=1}^{3} \frac{\partial u_h}{\partial x_i}\frac{\partial u_h}{\partial x_j}\right)(i, j = 1, 2, 3) \tag{3.6}$$

will be used as a local measure of *strain* (within eqn (3.6), x_1, x_2, x_3 stand for x, y, z and u_1, u_2, u_3 for u^P, v^P, w^P). By substituting eqns (3.5) into eqn (3.6), one readily obtains that all strain tensor components vanish identically, except ϵ_{11} and $\epsilon_{12} \equiv \epsilon_{21}$, whose values are given by

$$\epsilon_{11} = u' + \frac{1}{2}(u'^2 + v'^2) - y\phi'[(1 + u')\cos \phi + v' \sin \phi] + \frac{1}{2} y^2 \phi'^2 \tag{3.7^1}$$

$$2\epsilon_{12} = v' \cos \phi - (1 + u') \sin \phi \tag{3.7^2}$$

where a prime denotes differentiation with respect to s; note that ϵ_{12} is constant over a cross-section.

3.2 *Direct approach*

Following the treatment by Antman[10], we introduce the vector field (see Fig. 2)

$$\mathbf{b}(s) = -\sin \phi(s)\mathbf{i} + \cos \phi(s)\mathbf{j} \tag{3.8}$$

so that the pair $(\mathbf{a}, \mathbf{b})$ constitutes an orthonormal basis with the same orientation as $(\mathbf{i}, \mathbf{j})$ (recollect eqn (3.3)). The *strains* ϵ, γ, χ are defined through

$$\mathbf{r}' = (1 + \epsilon)\mathbf{a} + \gamma\mathbf{b} \tag{3.9^1}$$

$$\chi = \phi'. \tag{3.9^2}$$

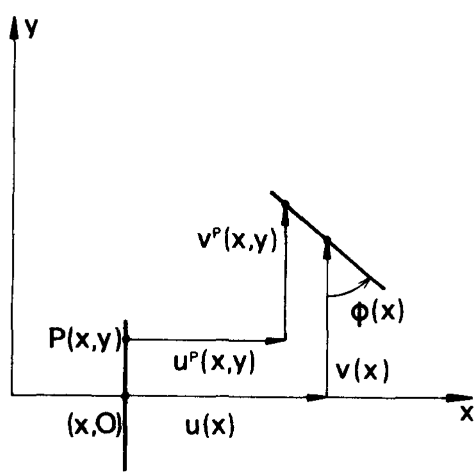

Fig. 1. Development from the three-dimensional theory.

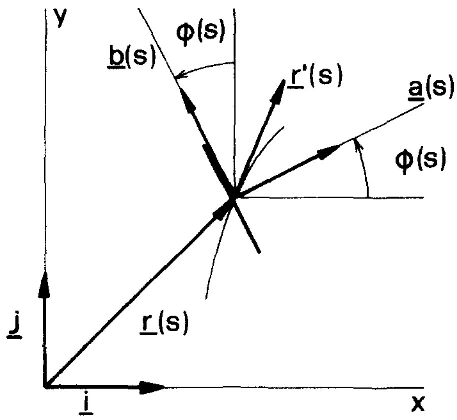

Fig. 2. Direct approach.

Equation (3.9¹) is tantamount to

$$\epsilon = (1 + u')\cos\phi + v'\sin\phi - 1 \qquad (3.10^1)$$

$$\gamma = v'\cos\phi - (1 + u')\sin\phi \qquad (3.10^2)$$

as it readily follows from eqn (3.2). Note that the right-hand sides of eqns (3.10²) and (3.7²) coincide.

It is perhaps worthy of mention that the same conclusions are arrived at by following a sort of "dual" argumentation, as the authors did in Ref. [11]. Stresses N, Q, M are introduced first, such that the internal force exerted across a cross-section is represented by

$$\mathbf{t} = N\mathbf{a} + Q\mathbf{b} \qquad (3.11)$$

and the internal couple by

$$\mathbf{m} = M\mathbf{a} \times \mathbf{b}. \qquad (3.12)$$

Equilibrium is then directly enforced in a generic configuration, and—via a rate of work equation—the strain rates associated to the above introduced stresses are expressed in terms of displacements and displacement rates:

$$\begin{aligned} \dot{\epsilon} &= \dot{\epsilon}(\mathbf{u}, \phi; \dot{\mathbf{u}}, \dot{\phi}) \\ \dot{\gamma} &= \dot{\gamma}(\mathbf{u}, \phi; \dot{\mathbf{u}}, \dot{\phi}) \\ \dot{\chi} &= \dot{\chi}(\mathbf{u}, \phi; \dot{\mathbf{u}}, \dot{\phi}). \end{aligned} \qquad (3.13)$$

Equations (3.13) turn out to be integrable, yielding exactly the same strain-displacement relationship represented through eqns (3.10), (3.9²).

3.3 *Constrained beam models*

From the exposition of the above two sub-sections, it is apparent that meaningful internal constraints, to which both beam models may conceivably be subjected, will in general be *nonlinear* by nature.

Consider the Bernoulli hypothesis: cross-sections remain normal to the line of centroids. Such a constraint is expressed in both models by the same equation

$$v'(s)\cos\phi(s) - [1 + u'(s)]\sin\phi(s) = 0, \ \forall s \in]0, \ell[\qquad (3.14)$$

which is easily satisfied by assuming that $\phi(\cdot)$ depends on $\mathbf{u}(\cdot)$ according to

$$\phi(s) = \arctan\frac{v'(s)}{1+u'(s)}. \tag{3.15}$$

However, linear *boundary* conditions on ϕ are transformed through eqn (3.15) into nonlinear ones in $\mathbf{u}'$.

Consider next the beam model of Sub-section 3.1 with an inextensible line of centroids; such a constraint is imposed through

$$u'(s)+\frac{1}{2}[u'^2(s)+v'^2(s)] = 0,\ \forall s \in]0, \ell[\tag{3.16}$$

or the beam model of Sub-section 3.2 subjected to the constraint $\epsilon = 0$, implying that

$$[1+u'(s)]\cos\phi(s)+v'(s)\sin\phi(s)-1 = 0,\ \forall s \in]0, \ell[. \tag{3.17}$$

Equations (3.16) and (3.17)—though different from each other—admit the same set of solutions *when coupled* with the shear undeformability constraint (3.14). This set is most straightforwardly represented by assuming that $\mathbf{u}(\cdot)$ depends on $\phi(\cdot)$ through

$$u'(s) = \cos\phi(s) - 1 \tag{3.18^1}$$

$$v'(s) = \sin\phi(s). \tag{3.18^2}$$

Again, troubles arise from kinematical boundary conditions. Boundary conditions will however be discussed in a proper variational setting within next section.

4. STRAIN ENERGY FUNCTIONAL FOR A FRAME

To describe a particular *hyper-elastic* beam, we need assign a specific form to the strain energy functional, that we shall express as

$$\Phi(u) \equiv \mathscr{E}(\epsilon(u)) \tag{4.1}$$

where $\epsilon(\cdot)$ represents the strain- displacement relationship. We shall consider but the simplest form of functional $\mathscr{E}(\cdot)$, assuming it to be *quadratic* and *homogeneous*:

$$\mathscr{E}(\alpha\epsilon) = \alpha^2\mathscr{E}(\epsilon),\ \forall \alpha \in \Re \tag{4.2}$$

(the second assumption is equivalent to the hypothesis that the considered reference configuration is stress-free).

Namely, for the beam model developed in Sub-section 3.1 we define the strain energy functional through

$$\mathscr{E}(\epsilon) \overset{\text{def}}{=} \int_{\mathscr{B}} \frac{1}{2}(E\epsilon_{11}^2 + 4G\epsilon_{12}^2)\,\mathrm{d}V \tag{4.3}$$

where $\mathscr{B} \subset \mathfrak{E}^3$ is the domain occupied by the beam in its reference configuration and E, G are three-dimensional elastic constants (under sufficiently strong hypotheses on material symmetry *any* functional (4.2) reduces to the form (4.3), on account that $\epsilon_{11}, \epsilon_{12} \equiv \epsilon_{21}$ are the sole non-vanishing strain components). Integral (4.3) is usefully evaluated as

$$\mathscr{E}(\epsilon) = \int_o^\ell \mathrm{d}s \int_{\mathscr{R}(s)} \frac{1}{2}(E\epsilon_{11}^2 + 4G\epsilon_{12}^2)\,\mathrm{d}A \tag{4.4}$$

$\mathscr{R}(s)$ being the two-dimensional domain occupied by the cross-section s. After substitution of eqns

(3.7) into eqn (4.4), the integration over $\mathcal{R}$ is easily performed. As a result, functional $\Phi(\cdot)$ will contain the following parameters: axial and shear rigidities EA, GA and flexural rigidities EI, EI_4; A being the area of $\mathcal{R}$, while

$$I \overset{\text{def}}{=} \int_{\mathcal{R}} y^2 \, \mathrm{d}A \tag{4.5^1}$$

$$I_4 \overset{\text{def}}{=} \int_{\mathcal{R}} y^4 \, \mathrm{d}A. \tag{4.5^2}$$

It is supposed that

$$\int_{\mathcal{R}} y \, \mathrm{d}A = 0 \qquad \int_{\mathcal{R}} y^3 \, \mathrm{d}A = 0. \tag{4.5^3}$$

Let us now consider the beam model developed in Sub-section 3.2; we shall pose for it

$$\mathscr{E}(\epsilon) \overset{\text{def}}{=} \int_o^{\ell} \frac{1}{2} (EA\epsilon^2 + GA\gamma^2 + EI\chi^2) \, \mathrm{d}s \tag{4.6}$$

where symbols EA, GA, EI for the elastic constants are borrowed from the previous model (material symmetry is again supposed to exclude coupling terms, bilinear in ϵ, γ, χ).

The material behaviour represented by eqn (4.4), together with eqns (3.7), is by no means identical to the one represented by eqns (4.6) and (3.9). We claim nothing about the possible physical foundation of any of the two models—which will henceforth be named Model 1 and 2, respectively. To the present purpose, it is sufficient to state that: (i) both models originate the same linearized problem (about the reference configuration), and (ii) for increasingly large axial and shear rigidities, both models consistently approach the inextensible, unshearable beam characterized by the strain energy functional

$$\Phi(u) \overset{\text{def}}{=} \int_o^{\ell} \frac{1}{2} EI\phi'^2 \, \mathrm{d}s \tag{4.7}$$

together with eqns (3.18).

In the next two sections, we shall indeed focus our attention on asymptotic solutions obtained for infinitely large EA and GA. This should not throw any doubt upon the usefulness of considering unconstrained beam models in the present context, however. As we strive to show, such a choice is in fact dictated by overriding computational reasons. By the way, we are fully aware that the analysis of *highly* axially and shear deformable beams does deserve attention (see, for instance, the brilliant qualitative analysis by Antman in [10], and, on the other side, the stimulating experimental evidence provided by Schapery and Skala in [12]). While the present computational approach could straightforwardly embody physically sound models of such phenomena, we keep this paper within a more limited scope, considering only the most unsophisticated constitutive relations.

The most commonly used "technical" beam models may be conceived as approximative versions of Model 1. We shall name Model 3 the one based on the assumptions:

$$\epsilon_{11} = u' + \frac{1}{2}(u'^2 + v'^2) - y\phi' \tag{4.8}$$

$$\phi = v' \tag{4.9}$$

$$\mathscr{E}(\epsilon) = \int_{\mathscr{B}} \frac{1}{2} \epsilon_{11}^2 \, \mathrm{d}V. \tag{4.10}$$

Equation (4.8)—though simplified with respect to eqn (3.7^1)—is still rigorously correct, being left invariant by an arbitrary *rigid* displacement from a generic configuration. On the contrary, eqn (4.9) implies the vanishing of shear deformation only to the first order (recollect eqn 3.15); hence the omission of shear strain energy from functional (4.10) induces the neglect of higher order contributions of shear *force* to the equilibrium equation (2.2). As we shall see in the following, such contributions may be not negligible, even for infinitely large shear (and axial) rigidities.

Often eqn (4.8) is substituted by

$$\epsilon_{11} = u' + \frac{1}{2} v'^2 - y\phi' \tag{4.11}$$

upon neglect of the term $(1/2)u'^2$, which is deemed somehow smaller than the remaining ones. It is easy to check that eqn (4.11) is no more invariant under rigid displacements. The model based on eqns (4.11), (4.9) and (4.10) will be called Model 4. In the next two sections, we shall see that—despite the "smallness" of the term $(1/2)u'^2$—Models 3 and 4 may give essentially *different* answers—though both erroneous.

We shall consider a *frame* as a collection of a number of the above described beams, whose ends are connected in a prescribed way at *joints*. We shall admit that for the frame under consideration a stress-free configuration exists, in which all beams are straight and have normal cross-sections; and we shall take this one as the reference configuration of the frame. The generic configuration of the ith beam will be identified through the functions $\mathbf{u}_i(\cdot)$, $\phi_i(\cdot)$ defined over the interval $[0, \ell_i]$, according to the treatment of Section 3. The strain energy functional for the entire frame is defined through

$$\Phi(u) \overset{\text{def}}{=} \sum_{i=1}^{N} \Phi_i(u_i) \tag{4.12}$$

where u_i abstractly refers to the ordered set of (sufficiently smooth) functions $(\mathbf{u}_i(\cdot), \phi_i(\cdot))$, belonging to a suitable product space $\mathcal{V}_i$; hence

$$u \in \mathcal{V} \overset{\text{def}}{=} \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_N. \tag{4.13}$$

Essential (i.e. kinematic) boundary conditions impose the displacement $\mathbf{u}$ and the rotation ϕ to be continuous across each joint (for the sake of simplicity, joint releases are not formally considered; nevertheless, their inclusion is trivial in concept). While obvious, it is of importance to distinguish between continuity of displacement $\mathbf{u}$ and continuity of displacement *components* u, v across a joint where differently directed axes meet.

For unconstrained beam models (or linearly constrained, such as Models 3 and 4) the above mentioned continuity conditions are linear; hence, the kinematically admissible subset $\mathcal{U} \subset \mathcal{V}$ is a linear manifold in $\mathcal{V}$. No more so for nonlinearly constrained beam models; while the configuration space $\mathcal{V}$ becomes smaller with respect to the corresponding unconstrained model, the kinematically admissible subset $\mathcal{U}$ get curved. The former circumstance is clearly advantageous, *per se*, but is largely outweighed by the penalty to be paid for the latter—in a computational context.

5. BUCKLING AND POST-BUCKLING ANALYSIS OF THE EULER STRUT

The present section is devoted to an analytical study of the classical Euler strut, depicted in Fig. 3(a), with the aim of gaining familiarity with the machinery developed in the previous sections.

A perturbation analysis of the inextensible, unshearable Euler strut is remarkably simple (see Refs. [1, 7–9]). The fundamental path is trivial, and exhibits the lowest bifurcation at $\lambda = \pi^2$, associated with the (simple) buckling mode

$$\dot{\phi}(s) = \cos(\pi s/\ell). \tag{5.1}$$

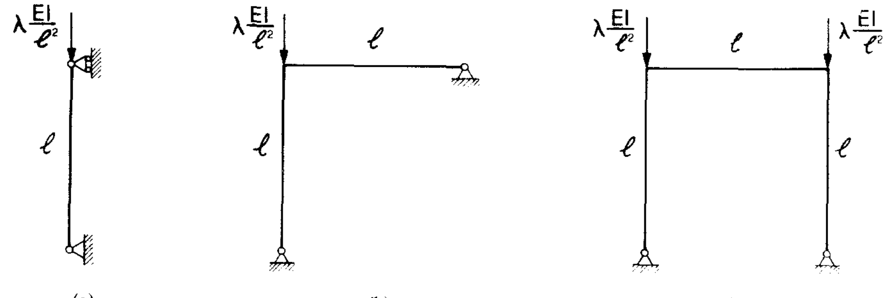

Fig. 3. (a) Euler strut, (b) Roorda frame, (c) Square hinged portal frame.

The identically null second order displacement rate

$$\ddot{\phi}(s) \equiv 0 \tag{5.2}$$

is associated with the following expansion of λ

$$\lambda = \pi^2\left(1 + \frac{1}{8}t^2 + \cdots\right) \tag{5.3}$$

where

$$t = \phi(0). \tag{5.4}$$

We intend here to compare these results with those obtained with Models 1 to 4, for infinitely large axial and shear rigidities. According to the treatment of Section 2, we shall need to compute the first four derivatives of functional $\Phi(\cdot)$; this being defined through eqn (4.1), the following relations will be employed:

$$\Phi'(u)u_1 = \mathscr{E}'(\epsilon(u))\epsilon'(u)u_1 \tag{5.5^1}$$

$$\Phi''_c u_1 u_2 = \mathscr{E}''_c \epsilon'_c u_1 \epsilon'_c u_2 + \mathscr{E}'_c \epsilon''_c u_1 u_2 \tag{5.5^2}$$

$$\begin{aligned}\Phi'''_c u_1 u_2 u_3 = {} & \mathscr{E}''_c \epsilon'_c u_1 \epsilon''_c u_2 u_3 + \mathscr{E}''_c \epsilon'_c u_2 \epsilon''_c u_3 u_1 \\ & + \mathscr{E}''_c \epsilon'_c u_3 \epsilon''_c u_1 u_2 + \mathscr{E}'_c \epsilon'''_c u_1 u_2 u_3\end{aligned} \tag{5.5^3}$$

$$\begin{aligned}\Phi_c^{IV} u_1 u_2 u_3 u_4 = {} & \mathscr{E}''_c \epsilon'_c u_1 \epsilon'''_c u_2 u_3 u_4 + \mathscr{E}''_c \epsilon'_c u_2 \epsilon'''_c u_3 u_4 u_1 \\ & + \mathscr{E}''_c \epsilon'_c u_3 \epsilon'''_c u_4 u_1 u_2 + \mathscr{E}''_c \epsilon''_c u_4 \epsilon'''_c u_1 u_2 u_3 \\ & + \mathscr{E}''_c \epsilon''_c u_4 u_1 \epsilon''_c u_2 u_3 + \mathscr{E}''_c \epsilon''_c u_4 u_2 \epsilon''_c u_3 u_1 \\ & + \mathscr{E}''_c \epsilon''_c u_4 u_3 \epsilon''_c u_1 u_2 + \mathscr{E}'_c \epsilon^{IV} u_1 u_2 u_3 u_4\end{aligned} \tag{5.5^4}$$

where, without loss of clarity, the same symbol (′) is being used to denote differentiation of each function $\Phi(\cdot)$, $\mathscr{E}(\cdot)$, $\epsilon(\cdot)$ with respect to its own argument. In deriving eqns (5.5^3), (5.5^4), account has already been taken of eqn (4.2), which implies that derivatives of functional $\mathscr{E}(\cdot)$ of order higher than two vanish identically.

5.1 *Model* 1

Along the fundamental path we shall consider in this sub-section—and the next ones–the beam undergoes *pure stretch*; this implies

$$\begin{aligned} v^f(s;\lambda) &\equiv 0 \\ \phi^f(s;\lambda) &\equiv 0. \end{aligned} \tag{5.6}$$

As one would expect, position (5.6) drastically simplifies our analysis, making in fact possible—together with the assumed homogeneity of the beam—a straightforward analytical treatment.

Equation (2.8^1), through eqns (5.5^1), (4.3) and (3.7), yields (for a purely stretched configuration)

$$EAu'(1+u')\left(1+\frac{1}{2}u'\right)+\lambda EI/\ell^2=0 \tag{5.7^1}$$

which implicitly describes

$$u^f(s;\lambda)\equiv u'(\lambda)s. \tag{5.7^2}$$

Equation (2.12), through eqn (5.5^2), yields the eigenvalue problem

$$EA\left(1+3u'+\frac{3}{2}u'^2\right)\dot{u}''=0 \tag{5.8^1}$$

$$GA[\dot{v}'-(1+u')\dot{\phi}]'+EAu'\left(1+\frac{1}{2}u'\right)\dot{v}''=0 \tag{5.8^2}$$

$$EI\left(1+3u'+\frac{3}{2}u'^2\right)\dot{\phi}''+GA(1+u')[\dot{v}'-(1+u')\dot{\phi}]=0 \tag{5.8^3}$$

equipped with boundary conditions

$$\dot{u}(0)=0,\qquad EA\left(1+3u'+\frac{3}{2}u'^2\right)\dot{u}'(\ell)=0 \tag{5.9^1}$$

$$\dot{v}(0)=\dot{v}(\ell)=0 \tag{5.9^2}$$

$$EI\left(1+3u'+\frac{3}{2}u'^2\right)\dot{\phi}'(0)=EI\left(1+3u'+\frac{3}{2}u'^2\right)\dot{\phi}'(\ell)=0. \tag{5.9^3}$$

Equations (5.8^1), (5.9^1) imply

$$\dot{u}_c(s)\equiv 0 \tag{5.10}$$

while eqns (5.8^2), (5.8^3) can be manipulated in order to give the standard equation

$$\dot{\phi}'''+k^2\dot{\phi}'=0 \tag{5.11}$$

where

$$(k\ell)^2=\frac{\lambda(1+u')^2}{\left(1+3u'+\frac{3}{2}u'^2\right)(1+u'-\lambda EI/GA\ell^2)}. \tag{5.12}$$

We will consider the eigenvalue λ_c corresponding to

$$k\ell=\pi. \tag{5.13}$$

It is easily seen that, when EA and GA tend to infinity,

$$\lambda_c\to\pi^2. \tag{5.14}$$

The associated buckling mode is described through

$$\dot{\phi}_c(s)=\cos(\pi s/\ell) \tag{5.15^1}$$

$$\dot{v}_c(s)=\frac{(1+u_c')^2}{1+u_c'-\lambda_c EI/GA\ell^2}\,\frac{\ell}{\pi}\sin(\pi s/\ell) \tag{5.15^2}$$

together with eqn (5.10). Notice that the buckling mode has been normalized so that

$$\dot{\phi}_c(0) = 1. \quad (5.16)$$

From eqn (2.15[1]), through eqn (5.5[3]), it readily follows that

$$\dot{\lambda}_c = 0 \quad (5.17)$$

while the second order displacement rate satisfying eqn (2.13), together with the normalization condition

$$\ddot{\phi}_c(0) = 0 \quad (5.18)$$

is characterized by

$$\ddot{u}'_c = -\left(1+3u'_c+\frac{3}{2}u'^2_c\right)^{-1}\left\{(1+u'_c)\dot{v}'^2_c+3\frac{EI}{EA}(1+u'_c)\dot{\phi}'^2_c-2\frac{GA}{EA}[\dot{v}'_c-(1+u'_c)\dot{\phi}_c]\dot{\phi}_c\right\} \quad (5.19^1)$$

$$\ddot{v}_c \equiv 0 \quad (5.19^2)$$

$$\ddot{\phi}_c \equiv 0 \quad (5.19^3)$$

(the expression of $\ddot{u}_c(s)$, being irrelevant to what follows, is not explicitly given). Owing to eqn (5.17), eqn (2.15[2]) simplifies to

$$\ddot{\lambda}_c = \frac{\Phi''_c\ddot{v}_c^2 - \frac{1}{3}\Phi_c^{IV}\dot{v}_c^4}{\Phi'''_c\hat{u}_c^f\dot{v}_c^2} \quad (5.20)$$

Through eqns (5.5[2]), (5.5[4]) the numerator of eqn (5.20) is evaluated as

$$\Phi''_c\ddot{v}_c^2 - \frac{1}{3}\Phi_c^{IV}\dot{v}_c^4 = \mathscr{E}''_c\epsilon'_c\ddot{v}_c\epsilon'_c\ddot{v}_c + \mathscr{E}'_c\epsilon''_c\ddot{v}_c^2 - \mathscr{E}''_c\epsilon''_c\dot{v}_c^2\epsilon''_c\dot{v}_c^2$$
$$-\frac{4}{3}\mathscr{E}''_c\epsilon'_c\dot{v}_c\epsilon'''_c\dot{v}_c^3 - \frac{1}{3}\mathscr{E}'_c\epsilon_c^{IV}\dot{v}_c^4. \quad (5.21)$$

Now, we have

$$\mathscr{E}''_c\epsilon''_c\ddot{v}_c\epsilon'_c\ddot{v}_c = \int_o^\ell EA(1+u'_c)^2\ddot{u}'^2_c\,\mathrm{d}s \quad (5.22^1)$$

$$\mathscr{E}'_c\epsilon''_c\ddot{v}_c^2 = \int_o^\ell EAu'_c\left(1+\frac{1}{2}u'_c\right)\ddot{u}'^2_c\,\mathrm{d}s \quad (5.22^2)$$

$$-\mathscr{E}''_c\epsilon''_c\dot{v}_c^2\epsilon''_c\dot{v}_c^2 = -\int_o^\ell\{EA\dot{v}'^4_c+2EI\dot{v}'^2_c\dot{\phi}'^2_c+EI_4\dot{\phi}'^4_c\}\,\mathrm{d}s \quad (5.22^3)$$

$$-\frac{4}{3}\mathscr{E}''_c\epsilon'_c\dot{v}_c\epsilon'''_c\dot{v}_c^3 = \int_o^\ell\{4EI(1+u'_c)^2\dot{\phi}_c^2\dot{\phi}'^2_c - 8EI(1+u'_c)\dot{\phi}_c\dot{v}'_c\dot{\phi}'^2_c$$
$$+4GA[\dot{v}'_c-(1+u'_c)\dot{\phi}_c]\dot{\phi}_c^2\dot{v}'_c$$
$$-\frac{4}{3}GA(1+u'_c)[\dot{v}'_c-(1+u'_c)\dot{\phi}_c]\dot{\phi}_c^3\Big\}\,\mathrm{d}s \quad (5.22^4)$$

$$-\frac{1}{3}\mathscr{E}'_c\epsilon_c^{IV}\dot{v}_c^4 = 0. \quad (5.22^5)$$

Particular attention should be paid to the circumstance that eqns (5.22[1]) and (5.22[3]) *diverge* when EA grows to infinity. Nevertheless, elementary calculations show that their sum remains finite. In the limit for $EA \to \infty$, $GA \to \infty$ we have in fact

$$\mathscr{E}''_c \epsilon'_c \ddot{v}_c \epsilon'_c \ddot{v}_c - \mathscr{E}''_c \epsilon''_c \dot{v}_c^2 \epsilon''_c \dot{v}_c^2 \to \int_o^\ell \left\{ -2\lambda_c \frac{EI}{\ell^2} \cos^4(\pi s/\ell) + 4\lambda_c \frac{EI}{\ell^2} \cos^2(\pi s/\ell) \sin^2(\pi s/\ell) \right\} \mathrm{d}s \qquad (5.23^1)$$

$$\mathscr{E}'_c \epsilon''_c \ddot{v}_c^2 \to - \int_o^\ell \lambda_c \frac{EI}{\ell^2} \cos^4(\pi s/\ell)\, \mathrm{d}s \qquad (5.23^2)$$

$$-\frac{4}{3} \mathscr{E}''_c \epsilon'_c \dot{v}_c \epsilon'''_c \dot{v}_c^3 \to \int_o^\ell \left\{ \frac{8}{3} \lambda_c \frac{EI}{\ell^2} \cos^4(\pi s/\ell) - 4\lambda_c \frac{EI}{\ell^2} \cos^2(\pi s/\ell) \sin^2(\pi s/\ell) \right\} \mathrm{d}s. \qquad (5.23^3)$$

Summing up, we obtain

$$\Phi''_c \ddot{v}_c^2 - \frac{1}{3} \Phi_c^{IV} \dot{v}_c^4 \to -\frac{1}{8} \lambda_c \frac{EI}{\ell}. \qquad (5.24)$$

It has to be mentioned that the limit (5.23[1]) is obtained upon the assumption that

$$\frac{EA\ell^2}{EI} \to \infty \Rightarrow \frac{EI_4}{EI\ell^2} \to 0 \qquad (5.25)$$

which seems reasonable indeed, implying

$$\rho_4/\ell = O(\rho/\ell) \qquad (5.26)$$

where

$$\rho \overset{\text{def}}{=} (I/A)^{1/2}, \qquad \rho_4 \overset{\text{def}}{=} (I_4/A)^{1/4} \qquad (5.27)$$

The denominator of eqn (5.20) is computed, according to eqn (5.5[3]), by summing the following contributions

$$\mathscr{E}''_c \epsilon'_c \hat{u}_c^f \epsilon''_c \dot{v}_c^2 = \int_o^\ell \{EA(1+u'_c)\hat{u}'_c \dot{v}'^2_c + EI(1+u'_c)\hat{u}'_c \dot{\phi}'^2_c\}\, \mathrm{d}s \qquad (5.28^1)$$

$$2\mathscr{E}''_c \epsilon'_c \dot{v}_c \epsilon''_c \hat{u}_c^f \dot{v}_c = 2\int_o^\ell \{EI(1+u'_c)\hat{u}'_c \dot{\phi}'^2_c - GA[\dot{v}'_c - (1+u'_c)\dot{\phi}_c]\hat{u}'_c \dot{\phi}_c\}\, \mathrm{d}s \qquad (5.28^2)$$

$$\mathscr{E}'_c \epsilon'''_c \dot{v}_c^2 \hat{u}_c^f = 0. \qquad (5.28^3)$$

In conclusion,

$$\Phi'''_c \hat{u}_c^f \dot{v}_c^2 \to -\frac{1}{2} \frac{EI}{\ell} \qquad (5.29)$$

and hence

$$\ddot{\lambda}_c \to \frac{1}{4} \lambda_c \qquad (5.30)$$

in accordance with eqn (5.3).

The presentation of next sub-sections, closely following the present one, is allowed to be more sketchy.

5.2 *Model* 2

The fundamental path is now linear (in λ):

$$EAu' + \lambda EI/\ell^2 = 0 \tag{5.31}$$

$$\Rightarrow u^f(s;\lambda) = -\lambda s EI/EA\ell^2. \tag{5.32}$$

The equations of critical equilibrium along this path are

$$EA\dot{u}'' = 0 \quad (5.33^1)$$

$$GA[\dot{v}' - (1+u')\dot{\phi}] + EAu'\dot{\phi}' = 0 \quad (5.33)^2$$

$$EI\dot{\phi}'' + [GA(1+u') - EAu'][\dot{v}' - (1+u')\dot{\phi}] = 0 \quad (5.33^2)$$

with boundary conditions '

$$\dot{u}(0) = 0, \qquad EA\dot{u}'(\ell) = 0 \quad (5.34^1)$$

$$\dot{v}(0) = \dot{v}(\ell) = 0 \quad (5.34^2)$$

$$EI\dot{\phi}'(0) = EI\dot{\phi}'(\ell) = 0. \quad (5.34^3)$$

Equations (5.33^2), (5.33^3) entail

$$\dot{\phi}''' + k^2\dot{\phi}' = 0 \tag{5.35}$$

with

$$(k\ell)^2 = \lambda(1 - \lambda EI/EA\ell^2 + \lambda EI/GA\ell^2). \tag{5.36}$$

By correspondence with

$$k\ell = \pi \tag{5.37}$$

the following buckling mode is obtained

$$\dot{u}_c(s) \equiv 0 \quad (5.38^1)$$

$$\dot{\phi}_c(s) = \cos(\pi s/\ell) \quad (5.38^2)$$

$$\dot{v}_c(s) = (1 - \lambda_c EI/EA\ell^2 + \lambda_c EI/GA\ell^2)\frac{\ell}{\pi}\sin(\pi s/\ell). \quad (5.38^3)$$

Subsequently, we obtain $\dot{\lambda}_c = 0$, and the second order displacement rate

$$\ddot{u}'_c = -(1 - 2\lambda_c EI/EA\ell^2 + 2\lambda_c EI/GA\ell^2)\dot{\phi}_c^2 \quad (5.39^1)$$

$$\ddot{v}_c \equiv 0 \quad (5.39^2)$$

$$\ddot{\phi}_c \equiv 0. \quad (5.39^3)$$

The ingredients of the numerator of eqn (5.20) are listed below:

$$\mathscr{E}''_c\epsilon'_c\ddot{v}_c\epsilon'_c\ddot{v}_c = \int_o^\ell EA\ddot{u}'^2_c\,\mathrm{d}s \quad (5.40^1)$$

$$\mathscr{E}'_c\epsilon''_c\ddot{v}_c^2 = 0 \quad (5.40^2)$$

$$-\mathscr{E}_c''\epsilon_c''\dot{v}_c^2\epsilon_c''\dot{v}_c^2 = -\int_o^\ell EA[2\dot{v}_c'\dot{\phi}_c - (1+u_c')\dot{\phi}_c^2]\,\mathrm{d}s \qquad (5.40^3)$$

$$-\frac{4}{3}\mathscr{E}_c''\epsilon_c'\dot{v}_c\epsilon_c'''\dot{v}_c^3 = \frac{4}{3}\int_o^\ell GA[\dot{v}_c' - (1+u_c')\dot{\phi}_c][3\dot{v}_c'\dot{\phi}_c - (1+u_c')\dot{\phi}_c^2]\,\mathrm{d}s \qquad (5.40^4)$$

$$-\frac{1}{3}\mathscr{E}_c'\epsilon_c^{IV}\dot{v}_c^4 = \frac{1}{3}\int_o^\ell EAu_c'[4\dot{v}_c'\dot{\phi}_c^3 - (1+u_c')\dot{\phi}_c^4]\,\mathrm{d}s. \qquad (5.40^5)$$

Each of eqns (5.40¹), (5.40³) diverges, in a way depending on *how EA* and *GA* approach infinity; nevertheless, the limit of their sum in finite and uniquely determined. We have in fact

$$\mathscr{E}_c''\epsilon_c'\ddot{v}_c\epsilon_c'\ddot{v}_c - \mathscr{E}_c''\epsilon_c''\dot{v}_c^2\epsilon_c''\dot{v}_c^2 \to -\int_o^\ell 2\lambda_c\frac{EI}{\ell^2}\cos^4(\pi s/\ell)\,\mathrm{d}s \qquad (5.41^1)$$

$$-\frac{4}{3}\mathscr{E}_c''\epsilon_c'\dot{v}_c\epsilon_c'''\dot{v}_c^3 \to \int_o^\ell \frac{8}{3}\lambda_c\frac{EI}{\ell^2}\cos^4(\pi s/\ell)\,\mathrm{d}s \qquad (5.41^2)$$

$$-\frac{1}{3}\mathscr{E}_c'\epsilon_c^{IV}\dot{v}_c^4 \to -\int_o^\ell \lambda_c\frac{EI}{\ell^2}\cos^4(\pi s/\ell)\,\mathrm{d}s. \qquad (5.41^3)$$

The denominator of eqn (5.20) is computed by summing the following contributions

$$\mathscr{E}_c''\epsilon_c'\hat{u}_c^f\epsilon_c''\dot{v}_c^2 = \int_o^\ell EA\hat{u}_c'[2\dot{v}_c'\dot{\phi}_c - (1+u_c')\dot{\phi}_c^2]\,\mathrm{d}s \to -\int_o^\ell \frac{EI}{\ell^2}\cos^2(\pi s/\ell)\,\mathrm{d}s \qquad (5.42^1)$$

$$2\mathscr{E}_c''\epsilon_c'\dot{v}_c\epsilon_c''\hat{u}_c^f\dot{v}_c = -2\int_o^\ell GA[\dot{v}_c' - (1+u_c')\dot{\phi}_c]\dot{\phi}_c\hat{u}_c'\,\mathrm{d}s \to 0 \qquad (5.42^2)$$

$$\mathscr{E}'\epsilon_c'''\dot{v}_c^2\hat{u}_c^f = -\int_o^\ell EAu_c'\hat{u}_c'\dot{\phi}_c^2\,\mathrm{d}s \to 0. \qquad (5.42^3)$$

In conclusion, we get once again

$$\ddot{\lambda}_c \to \frac{1}{4}\lambda_c \qquad (5.43)$$

5.3 *Model* 3

The fundamental path coincides with that of Model 1 (eqns 5.7), while the equations of critical equilibrium are

$$EA\left(1+3u'+\frac{3}{2}u'^2\right)\dot{u}'' = 0 \qquad (5.44^1)$$

$$EI\dot{v}^{IV} - EAu'\left(1+\frac{1}{2}u'\right)\dot{v}'' = 0 \qquad (5.44^2)$$

with boundary conditions

$$\dot{u}(0) = 0, \qquad EA\left(1+3u'+\frac{3}{2}u'^2\right)\dot{u}'(\ell) = 0 \qquad (5.45^1)$$

$$\dot{v}(0) = \dot{v}(\ell) = 0 \qquad (5.45^2)$$

$$EI\dot{v}''(0) = EI\dot{v}''(\ell) = 0. \qquad (5.45^3)$$

Equation (5.44²) may be rewritten in the standard form

$$\dot{v}^{IV} + k^2\dot{v}'' = 0 \qquad (5.46)$$

with

$$(k\ell)^2 = \frac{\lambda}{1+u'}. \qquad (5.47)$$

By correspondence with

$$k\ell = \pi \tag{5.48}$$

the following buckling mode is obtained

$$\dot{u}_c(s) \equiv 0 \tag{5.49^1}$$

$$\dot{v}_c(s) = \frac{\ell}{\pi}\sin(\pi s/\ell). \tag{5.49^2}$$

Subsequently, we get $\dot{\lambda}_c = 0$, and the second order displacement rate

$$\ddot{u}'_c = -\left(1 + 3u'_c + \frac{3}{2}u'^2_c\right)^{-1}(1 + u'_c)\dot{v}'^2_c \tag{5.50^1}$$

$$\ddot{v}'_c \equiv 0 \tag{5.50^2}$$

$$\ddot{\phi}'_c \equiv 0. \tag{5.50^3}$$

The numerator of eqn (5.20) is computed from

$$\mathscr{E}''_c\epsilon_c{}'\ddot{v}_c\epsilon'_c\ddot{v}_c = \int_o^\ell EA(1 + u'_c)^2\ddot{u}'^2_c\,\mathrm{d}s \tag{5.51^1}$$

$$\mathscr{E}'_c\epsilon''_c\ddot{v}_c^2 = \int_o^\ell EAu'_c\left(1 + \frac{1}{2}u'_c\right)\ddot{u}'^2_c\,\mathrm{d}s \tag{5.51^2}$$

$$-\mathscr{E}''_c\epsilon''_c\dot{v}_c^2\epsilon''_c\dot{v}_c^2 = -\int_o^\ell EA\dot{v}'^4_c\,\mathrm{d}s \tag{5.51^3}$$

$$-\frac{4}{3}\mathscr{E}''_c\epsilon'_c\dot{v}_c\epsilon'''_c\dot{v}_c^3 = 0 \tag{5.51^4}$$

$$-\frac{1}{3}\mathscr{E}'_c\epsilon_c{}^{IV}\dot{v}_c^4 = 0. \tag{5.51^5}$$

In the limit for EA and GA growing to infinity, we have

$$\mathscr{E}''_c\epsilon'_c\ddot{v}_c\epsilon'_c\ddot{v}_c - \mathscr{E}''_c\epsilon''_c\dot{v}_c^2 \rightarrow 2\int_o^\ell \lambda_c\frac{EI}{\ell^2}\cos^4(\pi s/\ell)\,\mathrm{d}s \tag{5.52^1}$$

$$\mathscr{E}'_c\epsilon''_c\ddot{v}_c^2 \rightarrow -\int_o^\ell \lambda_c\frac{EI}{\ell^2}\cos^4(\pi s/\ell)\,\mathrm{d}s. \tag{5.52^2}$$

The denominator of eqn (5.20) is computed from

$$\mathscr{E}''_c\epsilon'_c\hat{u}_c{}^f\epsilon''_c\dot{v}_c^2 = \int_o^\ell EA(1 + u'_c)\hat{u}'_c\dot{v}'^2_c\,\mathrm{d}s \rightarrow -\int_o^\ell \frac{EI}{\ell^2}\cos^2(\pi s/\ell)\,\mathrm{d}s \tag{5.53^1}$$

$$2\mathscr{E}''_c\epsilon'_c\dot{v}_c\epsilon''_c\hat{u}_c{}^f\dot{v}_c = 0 \tag{5.53^2}$$

$$\mathscr{E}'_c\epsilon'''_c\dot{v}_c^2\hat{u}_c^f = 0. \tag{5.53^3}$$

In conclusion, we obtain the *erroneous* result

$$\ddot{\lambda}_c \rightarrow -\frac{3}{4}\lambda_c. \tag{5.54}$$

5.4 *Model* 4

The fundamental path coincides with that of Model 2 (eqns (5.31), (5.32)), while the equations of critical equilibrium are

$$EA\dot{u}''=0 \tag{5.55^1}$$

$$EI\dot{v}^{IV}-EAu'\dot{v}''=0 \tag{5.55^2}$$

with boundary conditions

$$\dot{u}(0)=0,\quad EA\dot{u}'(\ell)=0 \tag{5.56^1}$$

$$\dot{v}(0)=\dot{v}(\ell)=0 \tag{5.56^2}$$

$$EI\dot{v}''(0)=EI\dot{v}''(\ell)=0. \tag{5.56^3}$$

Equation (5.55^2) is tantamount to

$$\dot{v}^{IV}+k^2\dot{v}''=0 \tag{5.57}$$

with

$$(k\ell)^2=\lambda. \tag{5.58}$$

By correspondence with

$$k\ell=\pi \tag{5.59}$$

the same buckling mode is obtained as with Model 3 (eqns (5.49)). Subsequently, we get $\dot{\lambda}_c=0$, and the second order displacement rate

$$\ddot{u}_c'=-\dot{v}_c'^2 \tag{5.60^1}$$

$$\ddot{v}_c\equiv 0 \tag{5.60^2}$$

$$\ddot{\phi}_c\equiv 0. \tag{5.60^3}$$

The numerator of eqn (5.20) is computed from

$$\mathscr{E}_c''\epsilon_c'\ddot{v}_c\epsilon_c'\ddot{v}_c=\int_o^\ell EA\ddot{u}_c'^2\,\mathrm{d}s \tag{5.61^1}$$

$$\mathscr{E}_c'\epsilon_c''\ddot{v}_c^{\,2}=0 \tag{5.61^2}$$

$$-\mathscr{E}_c''\epsilon_c''\dot{v}_c^{\,2}\epsilon_c''\dot{v}_c^{\,2}=-\int_o^\ell EA\dot{v}_c'^4\,\mathrm{d}s \tag{5.61^3}$$

$$-\frac{4}{3}\mathscr{E}_c''\epsilon_c'\dot{v}_c\epsilon_c'''\dot{v}_c^{\,3}=0 \tag{5.61^4}$$

$$-\frac{1}{3}\mathscr{E}_c'\epsilon_c^{IV}\dot{v}_c^{\,4}=0 \tag{5.61^5}$$

The sum of eqns (5.61^1), (5.61^3) *vanishes* for any value of *EA* (and *GA*); hence

$$\ddot{\lambda}_c=0 \tag{5.62}$$

6. A COMPARISON OF RESULTS FOR SOME TEST CASES

In this section a few quantitative results, relative to the problems sketched in Fig. 3, are collected and discussed. The early post-buckling behaviour of each system is characterized through the relation

$$\lambda/\lambda_c = 1 + \lambda_1 t + \lambda_2 t^2 \tag{6.1}$$

where

$$\lambda_1 \equiv \dot{\lambda}_c/\lambda_c \qquad (6.2^1)$$

$$\lambda_2 \equiv \frac{1}{2}\ddot{\lambda}_c/\lambda_c \qquad (6.2^2)$$

and t is a suitably chosen (problem-dependent) parameter. The following positions hold: for the Euler strut (Fig. 3a) t is identified with the end rotation; for the Roorda frame (Fig. 3b) with the clockwise joint rotation; and for the square hinged portal frame (Fig. 3c) with the ratio (beam sidesway)/(colums height).

As far as the bifurcation point and the initial slope of the post-buckling curve are concerned, *all* beam models considered in Section 4 give answers which are asymptotically coincident with the inextensible, unshearable beam model (characterized by strain energy (4.7)). Hence, a single list of values for λ_c and λ_1 is furnished in Table 1. On the contrary, Table 2 lists three different sets of values for the second-order coefficient λ_2. In fact, while Models 1 and 2—which are geometrically exact—give values of λ_2 which are asymptotically coincident with those obtained from the constrained model (first column of Table 2, labelled "Exact"), Models 3 and 4 fail to estimate correctly the *curvature* of the post-buckling curve at bifurcation (see second and third columns of Table 2).

The Euler strut has been extensively analysed in Section 5, where the relevant bibliographic information has also been given. The first analytical study of the Roorda frame is due to Koiter[14], who employed the theory we have called Model 3. Roorda and Chilver[15] solved the same problem by using the purely flexible beam model. The treatment by Koiter is essentially duplicated within the book by Brush and Almroth[16], in terms of what we call Model 4. All of these authors obtain coincident results for λ_c and λ_1. None of them pursue the evaluation of λ_2, which has been computed by Di Carlo *et al.*[17] using the purely flexible beam

Table 1. Coefficients λ_c and λ_1

	λ_c	λ_1
Euler strut	9.87	0.
Roorda frame	13.89	.380
Portal frame	1.82	0.

Table 2. Coefficient λ_2

	Exact	Model 3	Model 4
Euler strut	.125	-.375	0.
Roorda frame	.464	-.631	.227
Portal frame	.158	-.803	-.081

model (more detailed information may be found in Ref. [18]). The symmetric hinged portal frame has been analysed in Refs. [17, 19] on the base of the purely flexible beam model. Model 2 was previously employed for the same problem by Alessi *et al.*[20].

The performance of exact vs geometrically approximate beam models displayed by Tables 1 and 2 is obviously far from accidental. A scrutiny of the perturbation procedure sketched in Section 2, and detailed on a sample problem in Section 5, shows that, *if the fundamental state is purely stretched*, the kinematic approximations introduced with Models 3 and 4 induce errors which are vanishingly small—when the axial rigidity grows to infinity—up to, and including, eqn (2.13), which determines the second-order displacement rate $\ddot{v}_c$. No more so for the eqn (2.14), determining $\dddot{v}_c$—and hence $\ddot{\lambda}_c$. The above statement remains true if in the fundamental state the beam is *asymptotically* axially loaded—as it happens to be for the Roorda frame.

As a consequence, the analysis of the Roorda frame in Refs. [14, 16] is perfectly legitimate, being pushed not beyond third-order energy terms. On the contrary, the use of Model 4 (or 3) for analysing fourth-order systems—as proposed for instance by Care *et al.*[21]—is devoid of foundation. The same criticism applies, *a fortiori*, to global nonlinear analyses based on kinematically approximate beam models. It is difficult, for instance, to assess the *real* accuracy of the direct non-linear analysis of the Roorda frame by Kounadis *et al.*[22], based on Model 4, in spite of the extreme *numerical* accuracy of the alleged results.

7. A COMMENT ON FINITE ELEMENT REPRESENTATION OF NONLINEAR BEAM MODELS

As a conclusion, a brief comment on the actual numerical implementation of the described perturbation procedure and nonlinear beam models, seems to be appropriate. We shall concentrate on *compatible* finite element models of geometrically exact theories of unconstrained planar beams.

Non-linearly constrained models have in fact been shown to be unsuitable for the analysis of *general* planar frames. Within Section 1, we have already cited the work by Thompson and Hunt[8], as an example of application of the finite element technique, which—while successful in the simple problems considered by these authors—has no capabilities as a "general purpose" procedure. It is evident, in fact, that within the formulation adopted by Thompson and Hunt (purely flexible beam with the transverse displacement component $v(s)$ as the sole field variable), such a basic operation as the enforcement of displacement continuity across a joint requires non-trivial *ad hoc* programming. One could hopefully expect that releasing the sole axial inextensibility constraint would be enough for getting off troubles. A glance at eqn (3.15) should, however, convince of the contrary. In fact, if Bernoulli hypothesis holds, the only practical way of enforcing the continuity of rotation across a joint, is to require the gradients of *both* displacement components u', v' to be *separately* continuous across the joint. Unfortunately, the solution itself can not possess such a continuity, wherever a geometric, or dynamic, or material singularity exists. Actually, this approach has been fruitfully adopted for performing nonlinear finite element analyses of single homogeneous beams loaded at the ends[23]. Its application to slightly more complicated frame problems would exhibit disastrous convergence properties.

On the contrary, unconstrained—though geometrically exact—beam models, such as Models 1 and 2, lend themselves naturally to the implementation of a simple and efficient stiffness method, based on the same standard interpolation functions currently used in linear analyses (accounting for shear deformability). The results presented in Tables 1 and 2 have in fact been obtained through parallel analytical and numerical procedures. A completely satisfactory agreement has been found—both for buckling and post-buckling—with a reasonably small number of freedoms.

We defer to a specific paper the detailed description of this finite element perturbation technique (but an early account may be found in Ref. [11]). A specific difficulty is however appropriately mentioned here, being connected with the treatment of axially and/or transversely *stiff* beams (to fix ideas, think of ratios $EI/EA\ell^2$, $EI/GA\ell^2$ or order 10^{-5}).

It is well known that rigidities different by orders of magnitude may cause numerical troubles due to truncation errors, also in linear analyses. Apart from that, a specific problem arises in the computation of the second order load parameter rate $\ddot{\lambda}_c$, which is directly connected with the finite element representation, and not with the implementation on digital computer. We have seen in Section 5 that the dominant contributions to the numerator of eqn (2.15^2) yielding $\ddot{\lambda}_c$, are

$$\int_o^{\ell} EA\ddot{u}_c'^2\,ds \tag{7.1}$$

and

$$-\int_o^{\ell} EA\dot{\phi}_c^4\,ds. \tag{7.2}$$

But terms (7.1) and (7.2) nearly cancel each other, and the numerator of eqn (2.15^2) turns out to be of the order of EI/ℓ. This circumstance is not peculiar to the particular problem treated in Section 5; on the contrary, it occurs in any post-buckling analysis, possibly in a more complicated form. The simple explanation is that for the inextensible, unshearable beam we should have exactly

$$\ddot{u}_c' = -\dot{\phi}_c^2 \tag{7.3}$$

and hence axial and shear rigidities growing to infinity put an increasingly large penalty on any solution that violates eqn (7.3), even slightly.

The consequences are that if the finite element representation of $\ddot{u}_c'$ doesn't match with that of $\dot{\phi}_c^2$, the numerical value obtained for $\ddot{\lambda}_c$ will be dominated by the interpolation error, magnified by a large elastic constant. Of course, the standard interpolation ($u(\cdot)$ elementwise linear, $\phi(\cdot)$ quadratic, $v(\cdot)$ cubic), while excellent for representing the fundamental path and the buckling mode, can not cope with eqn (7.3), a piecewise constant $\ddot{u}_c'$ being confronted with a piecewise quartic $\dot{\phi}_c^2$. Different avenues are open for giving this problem a computationally sound solution: reduced integration, strain interpolation, addition of special "bubble" functions. The last two alternatives have been experimented with full success.

REFERENCES

1. W. T. Koiter, *Over de Stabiliteit van het Elastisch Evenwicht.* Thesis, Delft, H. J. Paris, Amsterdam (1945). First English translation issued as *NASA TT F*-10, 833 (1967).
2. J. T. Oden and K. J. Bathe, A commentary on computational mechanics. *AMR* **31**, 1053 (1978).
3. R. H. Gallagher, Perturbation procedures in nonlinear finite element structural analysis. In *Computational Mechanics* (Edited by J. T. Oden) Lecture Notes in Mathematics **461**, Springer Verlag, Berlin (1975).
4. R. Casciaro, A. Di Carlo and M. Pignataro, Towards an efficient F. E. procedure for elastic bifurcation analysis. To be published in the *Int. J. Num. Meth. Engng.*
5. S. S. Antman, The theory of rods. In *Encyclopedia of Physics*, Vol. VIa/2 (*Mechanics of Solids II*) (Edited by C. Truesdell), Springer Verlag, Berlin (1972).
6. W. T. Koiter, General equations of elastic stability for thin shells. In *Proc. Symp. on the Theory of Shells to Honor Lloyd Hamilton Donnell* (Edited by D. Muster), University of Houston, Houston, Texas (1967).
7. S. J. Britvec, *The Stability of Elastic Systems*, Pergamon Press, New York (1973).
8. J. M. T. Thompson and G. W. Hunt, *A General Theory of Elastic Stability.* Wiley, London (1973).
9. B. Budiansky, Theory of buckling and post-buckling behaviour of elastic structures. In *Advances in Applied Mechanics* (Edited by Chia-Shun Yih), Vol. 14. Academic Press, New York (1974).
10. S. S. Antman, Bifurcation problems for nonlinearly elastic structures. In *Applications of Bifurcation Theory* (Edited by P. H. Rabinowitz), Academic Press, New York (1977).
11. R. Casciaro, A. Di Carlo and M. Pignataro, A finite element technique for bifurcation analysis. *Report II—192 Istituto di Scienza delle Costruzioni*, University of Rome (1976).
12. R. A. Schapery and D. P. Skala, Elastic stability of laminated elastomeric columns. *Int. J. Solids Structures* **12**, 401 (1976).
13. A. Di Carlo, M. Pignataro and N. Rizzi, On the proper treatment of axial and shear undeformability constraints in post-buckling analysis of beams. To be published in the *Int. J. Non-Linear Mech.*
14. W. T. Koiter, Post-buckling analysis of a simple two-bar frame. In *Recent Progress in Applied Mechanics*, The Folke Odqvist Volume. Almqvist & Wiksell, Stockholm (1967).
15. J. Roorda and A. H. Chilver, Frame buckling: an illustration of the perturbation technique. *Int. J. Non-Linear Mech.* **5**, 235 (1970).
16. D. O. Brush and B. O. Almroth, *Buckling of Bars, Plates and Shells.* McGraw-Hill, New York (1975).
17. A. Di Carlo, M. Pignataro and P. D'Asdia, A modified potential energy approach to post-buckling analysis of frames. *ZAMM* **58**, T 137 (1978).

18. N. Rizzi, A. Di Carlo and M. Pignataro, A parametric post-buckling analysis of an asymmetric two-bar frame, *J. Struct. Mech.* **8**, 435 (1980).
19. M. Pignataro, N. Rizzi and A. Di Carlo, Symmetric bifurcation of plane frames through a modified potential energy approach, *J. Struct. Mech.* **8**, 237 (1980).
20. M. Alessi, P. D'Asdia and M. Pignataro, Post-buckling analysis of some sample frames. *Report II—189 Istituto di Scienza delle Costruzioni*, University of Rome (1976).
21. R. F. Care, R. E. Lawther and A. P. Kabaila, Finite element post-buckling analysis for frames. *Int. J. Num. Meth. Engng* **11**, 833 (1977).
22. A. N. Kounadis, J. Giri and G. J. Simitses, Nonlinear stability analysis of an eccentrically loaded two-bar frame. *J. Appl. Mech.* **44**, 701 (1977).
23. M. Epstein and D. W. Murray, Large deformation in-plane analysis of elastic beams. *Comput. Structures* **6**, 1 (1976).